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Mixing exponential method and Toda lattice

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Abstract : In an attempt to find a general recipe for obtaining single solitary wave solutions, a modified mixing exponential approach is presented to deal with the discrete models. This requires establishment of new mathematical identities. The necessary identity for Toda system is established and all the possible solitary waves are obtained in a series form. With the proper choice of parameters the series solution reduces to all known single solitary wave solutions. Again the important case of diatomic Toda system is analysed with the help of mixing exponential approach and approximate solutions are discussed in the light of KAM theorem. An unexpected result is obtained which suggests only acoustic solitons.

Keywords : Toda lattice, mixing exponential method, solitons and nonlinear lattices

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1. Introduction

For quite a few years, a physical approach better known as mixing exponential method has been used in the literature [1] for constructing solitary wave solutions of a number of non-linear equations having application in a variety of physical systems. Conceptually, the method is very appealing as it starts with the real exponential travelling wave solutions of the underlying linear equations. The presence of non-linearity suggests a mixing of these exponentials and in the method, one thus seeks a solution to the nonlinear equations in the form of a series involving all the mixing exponentials. The method has some similarity with the previous works of Korpel [2]. There also exists a fairly good resemblance between the real exponential approach and the Fourier approach used by Dash and Patnaik [3], for the construction of periodic solutions, because solitary solution can be obtained from the cnoidal periodic solution in the long wavelength limit. In comparision with other methods like IST, Hirota's bilinear method, Bäcklund transformation *etc.*, this direct method is easily accessible and in addition, it introduces a new dimension to the non-linear theory by stressing the fundamental importance of the exponential solutions of the underlying linear equations. Using this direct approach, Hereman *et al* [1] analysed a fairly large number of non-linear equations and outlined different steps for obtaining solitary wave solutions. All these works refer to continuous cases including system of partial differential equations. Its application to discrete difference equations is long overdue and no such case has been analysed till date because it requires intricate mathematical modifications and also the derivation of new mathematical identities. Of all the discrete lattice equations, the only completely integrable equation is the Toda lattice equation [4]. So the purpose of this paper is to examine Toda lattice with a view to extending 'the mixing exponential method to differential difference equations. We observe that the discrete Toda problem needs certain modifications in the procedure as well as establishment of a crucial mathematical identity. Here, we intend to establish the identity and apply this real exponential method to a discrete system for the first time. Further in this paper, we want to analyse diatomic Toda equations with the help of this method.

2. Theory

(a) Toda lattice :

The equation of motion for one dimensional lattice with nearest neighbour interaction potential of the form

$$\phi(r) = \exp(-r) + r \tag{1}$$

can be written as

$$\ddot{y}_n = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}), \tag{2}$$

where y_n denotes the displacement of the n-th unit mass. Changing variables as in Toda 1975, eq. (2) can be recasted as

$$\ddot{s}_n = (1 + \dot{s}_n)(s_{n+1} + s_{n-1} - 2s_n), \tag{3}$$

where $$\dot{s}_n = ds_n / dt = \exp(y_{n-1} - y_n) - 1. \tag{4}$$

Linear part of eq. (3) is

$$\ddot{s}_n = s_{n+1} + s_{n-1} - 2s_n \tag{5}$$

and it admits growing and decaying exponential solutions of the form exp ($\pm KZ$) where K is an arbitray positive constant and

$$Z = An - Bt, \text{ provided}$$

$$KB = \pm 2 \sinh (KA/2). \tag{6}$$

We may observe that if s_n is a solution then $(s_n + c)$ is also a solution for any constant c. Now abbreviating the decaying exponential, exp ($-KZ$) by g, we seek a solution to eq. (3) in the form of power series in g,

$$s_n = \sum_{j=1}^{\infty} a_j g^j$$

substitution of the expression for s_n in (3) yields

$$\begin{aligned} s_n &= \sum_{j=1}^{\infty} a_j g^j \left\{K^2B^2j^2 - [\exp(jKA)+\exp(-jKA)-2]\right\} \\ &= KB\sum_{p=1}^{\infty} pa_p g^p \sum_{j=1}^{\infty} a_j g^j [\exp(jKA)+\exp(-jKA)-2] \\ &= KB\sum_{j=2}^{\infty}\sum_{p=1}^{j-1} (j-p)a_p a_{j-p} g^j [\exp(pKA)+\exp(-pKA)-2]. \end{aligned} \tag{7}$$

So, the recursion relation becomes

$$\begin{aligned} &\left(K^2B^2j^2 - [\exp(jKA)+\exp(-jKA)-2]\right)a_j \\ &= KB\sum_{p=1}^{j-1} (j-p)a_p a_{j-p} [\exp(pKA)+\exp(-pKA)-2] \end{aligned} \tag{8}$$

for $j \geq 2$ and here a_1 is arbitrary.

To solve this recursion relation we require the following mathematical identity, which can be proved by induction (since this identity is very crucial we establish it in the appendix)

$$\begin{aligned} &\sum_{p=1}^{j-1} (j-p)[\exp(pKX)+\exp(-pKX)-2] \\ &= -j^2 + [\exp(jKX)+\exp(-jKX)-2][\exp(KX)+\exp(-KX)-2]^{-1} \end{aligned} \tag{9}$$

For $j = 2$, eq. (8) becomes,

$$\begin{aligned} &\left\{4K^2B^2 - [\exp(2KA)+\exp(-2KA)-2]\right\}a_2 \\ &= KBa_1^2[\exp(KA)+\exp(-KA)-2] \end{aligned} \tag{10}$$

From identity (9) with $p = 1$ and $j = 2$, we have

$$\begin{aligned} &\exp(KA)+\exp(-KA)-2 \\ &= -4 + [(\exp(2KA)+\exp(-2KA)-2)][\exp(KA)+\exp(-KA)-2]^{-1}. \end{aligned}$$

Using the above expression as well as (6) in eq. (10), we obtain

$$\begin{aligned} &\left\{4K^2B^2 - [\exp(2KA)+\exp(-2KA)-2]\right\}a_2 \\ &= KBa_1^2(-4 + [\exp(2KA)+\exp(-2KA)-2]/K^2B^2). \end{aligned}$$

So,

$$a_2 = KB(-a_1^2 / K^2B^2)$$
$$= (-1)^2(-KB)(a_1 / KB)^2 \tag{11}$$

Similarly for $j = 3$, recursion relation (8) gives

$$\{9K^2B^2 - [\exp(3KA) + \exp(-3KA) - 2]\}a_3$$
$$= KBa_1a_2 \sum_{p=1}^{2}(3-p)[\exp(pKA) + \exp(-pKA) - 2]. \tag{12}$$

From the identity (9), we obtain

$$\sum_{p=1}^{2}(3-p)[\exp(PKA) + \exp(-PKA) - 2]$$
$$= -9 + [\exp(3KA) + \exp(-3KA) - 2][\exp(KA) + \exp(-KA) - 2]^{-1}.$$

Using the above expression as well as eq. (6) in eq. (12)

$$\{9K^2B^2 - [\exp(3KA) + \exp(-3KA) - 2]\}a_3$$
$$= KBa_1a_2(-K^2B^2)^{-1}(9K^2B^2 - [\exp(3KA) + \exp(-3KA) - 2]).$$

So,

$$a_3 = (-1)^3(-KB)(a_1 / KB)^3. \tag{13}$$

From eqs. (11) and (13) general form, a_p becomes transparent.

$$a_p = (-KB)(-a_1 / KB)^P. \tag{14}$$

Similarly, considering increasing exponentials exp (KZ) as g, the recursion relation yields

$$a_p = (KB)(a_1 / KB)^p. \tag{15}$$

(b) *Construction of solutions of monatomic Toda lattice :*

(a) In case of decreasing exponentials,

$$g = \exp(-KZ).$$

For $\quad a_1 > 0, a_p = -KBb^p(-1)^p$

With $\quad b_1 = a_1 / KB$

$$s_n = -KB\sum_{p=1}^{\infty}(-1)^p b^p g^p$$
$$= -KB(-bg + b^2g^2 - \ldots\ldots), \tag{16}$$
$$= -KB[-bg(1 - bg + b^2g^2 - \ldots\ldots)],$$
$$= KB[bg / (1 + bg)].$$

Writing b as exp $(-\delta)$ and $bg = \exp(-x)$ where x stands for $(KZ + \delta)$ and eq. (16) will be convergent for $x = KZ + \delta > 0$

$$s_n = \pm\sinh(KA/2)(1 - \tanh\frac{1}{2}[K(An - Bt) + \delta]). \tag{17}$$

For $$a_1 < 0, a_p = -KBb^p \text{ with } b = |a_1|/KB,$$

$$s_n = -KB\sum_{p=1}^{\infty} b^p g^p$$

Now, again taking $b = \exp(\delta)$,

s_n will represent a convergent series for $x > 0$ so that

$$s_n = \pm\sinh(KA/2)[1 - \coth\frac{1}{2}(K(An - Bt) + \delta)]. \tag{18}$$

(b) For growing exponentials $g = \exp(KZ)$,

$$s_n = \sum_{p=1}^{\infty} a_p g^p, a_p = KB(a_1/KB)^p$$

When $a_1 > 0$, taking $(a_1/KB) = b$,

$$s_n = KB[bg/(1 - bg)] \text{ for } x < 0,$$

with $bg = \exp(x)$, so that

$$s_n = \pm\sinh(KA/2)[1 - \coth[\frac{1}{2}(K(An - Bt) + \delta)]. \tag{19}$$

When $a_1 < 0$,

$$s_n = \pm\sinh(KA/2)[1 - \tanh[\frac{1}{2}(K(An - Bt) + \delta)]. \tag{20}$$

The eqs. (17–20) represent convergent solutions in the entire region $-\infty < z < +\infty$ and their solutions can be written as

$$s_n = \pm\sinh\alpha[1 - \tanh(\alpha n - \beta t + \varepsilon)] \tag{21}$$

and

$$s_n = \pm\sinh\alpha[1 - \coth(\alpha n - \beta t + \varepsilon)], \tag{22}$$

where $\alpha = (KA/2)$, $\beta = (KB/2)$ and ε is a constant phase $= \delta/2$.

3. Diatomic Toda lattice

Amongst all nonlinear lattices Toda lattice occupies a special position, being the only completely integrable nonlinear lattice. So, many attempts are made to study its diatomic version [3,5–7]. Numerical work of Casati and Ford [5] and Painleve analysis of Bountis *et al* [6] show that diatomic Toda represents a nonintegrable lattice. But the dynamic form factor calculation of Diederich [7] suggests soliton type solutions. Again KAM theorem implies atleast solutions very near to the integrable case. We here apply mixing exponential method to shed some light on diatomic Toda case.

Diatomic Toda lattice equations can be written as

$$-m_1 m_2 \ddot{s}_{2n} + m_1 s_{2n-1} + m_2 s_{2n+1} - (m_1 + m_2) s_{2n}$$
$$= \dot{s}_{2n}\left[(m_1 + m_2) s_{2n} - m_1 s_{2n-1} - m_2 s_{2n+1}\right], \quad (23)$$

$$-m_1 m_2 \ddot{s}_{2n-1} + m_1 s_{2n} + m_2 s_{2n-2} - (m_1 + m_2) s_{2n-1}$$
$$= \dot{s}_{2n-1}\left[(m_1 + m_2) s_{2n-1} - m_2 s_{2n-2} - m_1 s_{2n}\right]. \quad (24)$$

Let
$$s_{2n} = \sum_{j=1}^{\infty} a_j g_{2n}^j, g_{2n} = \exp(KZ), Z = (2nA - Bt) \quad (25)$$

and
$$s_{2n-1} = \sum_{p=1}^{\infty} b_p g_{2n-1}^p, g^{2n-1} = \exp(KY), Y = (2n-1)A - Bt, \quad (26)$$

where a_j, b_p are running coefficients and A, B are travelling wave parameters. Linearized equations corresponding to eqs. (23) and (24) can have exponential solutions, if

$$(m_1 + m_2 + m_1 m_2 K^2 B^2)^2 = [m_1 \exp(-KA) + m_2 \exp(+KA)]$$
$$[m_1 \exp(KA) + m_2 \exp(-KA)]. \quad (27)$$

Now coefficients a_1 and b_1 can be found out to be

$a_1 = b_1 L(K) X^{-1}(K)$ with a_1 arbitrary. Eqs. (23) and (24) also yield following recursion relations for $j \geq 2$.

$$-a_j (K^2 B^2 j^2 m_1 m_2 + m_1 + m_2) + b_j X(-jK)$$
$$= KB \sum_{p=1}^{j-1} (j-p) a_{j-p}\left[-(m_1 + m_2) a_p + b_p X(-pK)\right] \quad (28)$$

$$a_j X(jK) - b_j (K^2 B^2 j^2 m_1 m_2 + m_1 + m_2)$$
$$= KB \sum_{p=1}^{j-1} (j-p) b_{j-p}\left[-(m_1 + m_2) b_p + a_p X(+pK)\right], \quad (29)$$

where $X(jK) = m_1 \exp(jKA) + m_2 \exp(-jKA)$

and $L(nk) = (nk)^2 B^2 m_1 m_2 + m_1 + m_2$ with $n = 1, 2, 3 \ldots$.

For $j = 2$, $p = 1$ and a_2, b_2 can be obtained as

$$a_2 = K^3 B^3 m_1 m_2 . a_1^2 \left[X(2K).X(-2K) - L^2(2K)\right]^{-}$$
$$. \times \left[X(K).X^{-1}(-K).X(-2K) + L(2K)\right], \quad (30)$$

$$b_2 = K^3B^3m_1m_2.b_1^2\left[X(2K).X(-2K)-L^2(2K)\right]^{-1}$$

$$\times\,[X(-K).X^{-1}(K)X(2K)+L]. \tag{31}$$

Similarly for $j = 3, p = 1, 2$ and

$$a_3 = [a_1^3ML(3K)+b_1^3NX(-3K)]KBG, \tag{32}$$

$$b_3 = KBG[X(3K).M.a_1^3+L(3K).N.b_1^3], \tag{33}$$

where $\quad G^{-1} = X(-3K).X(3K)-L^2(3K);$

$$MH = [X(K).X^{-1}(-K).X(-2K)]+L(2K)]I+[X(-2K)X^{-2}(-K)L^2(K)]$$

$$[X(-K.)X^{-1}(K).X(2K)+L(2K)$$

with $\quad I = 2K^2B^2m_1m_2-m_1-m_2$

and $\quad H = (K^3B^3m_1m_2)^{-1}[X(2K).X(-2K)-L^2(2K);$

$$NH = [X(-K).X^{-1}(K).X(2K)+L(2K)]I$$

$$+X(2K).X^2(K).L^{-2}(K)[X(K).X^{-1}(-K)X(-2K)+L(2K)].$$

Solution can be written as

$$s_{2n} = a_1\exp(KZ)+a_2\exp(2KZ)+a_3\exp(3KZ)+\ldots \tag{34}$$

and

$$s_{2n-1}^{\cdot} = b_1\exp(KY)+b_2\exp(2KY)+b_3\exp(3KY)+\ldots \tag{35}$$

Solutions (34) and (35) may converge in the regions $Y, Z < 0$. Again considering decaying exponential solutions it can be shown that the solutions may converge in the region $Y, Z > 0$. So (34) and (35) represents solutions in the entire region.

For $m_1 = m_2 = 1$, $a_1 = b_1$ = arbitrary; $a_2 = b_2 = a_1^2 / KB$

$a_3 = b_3 = (a^3/K^2B^2)$ and the solutions (34) and (35) reduce to monatomic case.

The series solutions with coefficients (30–33) will represent soliton like solutions provided the coefficients converge. It is not easy to establish the convergence mathematically. So we examine two limiting situations. From *KAM* theorem it is obvious that for small perturbations trajectories near equal mass solution may be possible. Hence, first we establish that when $m_1 = m_2$, the coefficients reduce to monatomic Toda case. Secondly, for diatomic Toda case, choosing $m_1 = 1$, $m_2 = 2$, $k = 2$ and $B = 1$ we obtain A = 1.015036403 from (27), with these numerical values (34) and (35) yield

$$s_{2n} = 2\exp(-2z)+1.10\exp(-4z)+0.59\exp(-6z), \tag{36}$$

$$s_{2n+1} = 0.188 \exp(-2z) + 0.015 \exp(-4z) + 1.310 \times 10^{-3} \exp(-6z). \quad (37)$$

Their behaviour is illustrated by plotting graphs for $Z \geq 0.8$ in Figures (1 and 2) obviously in half plane only. The purpose here is very clear. We know that the approximate solutions

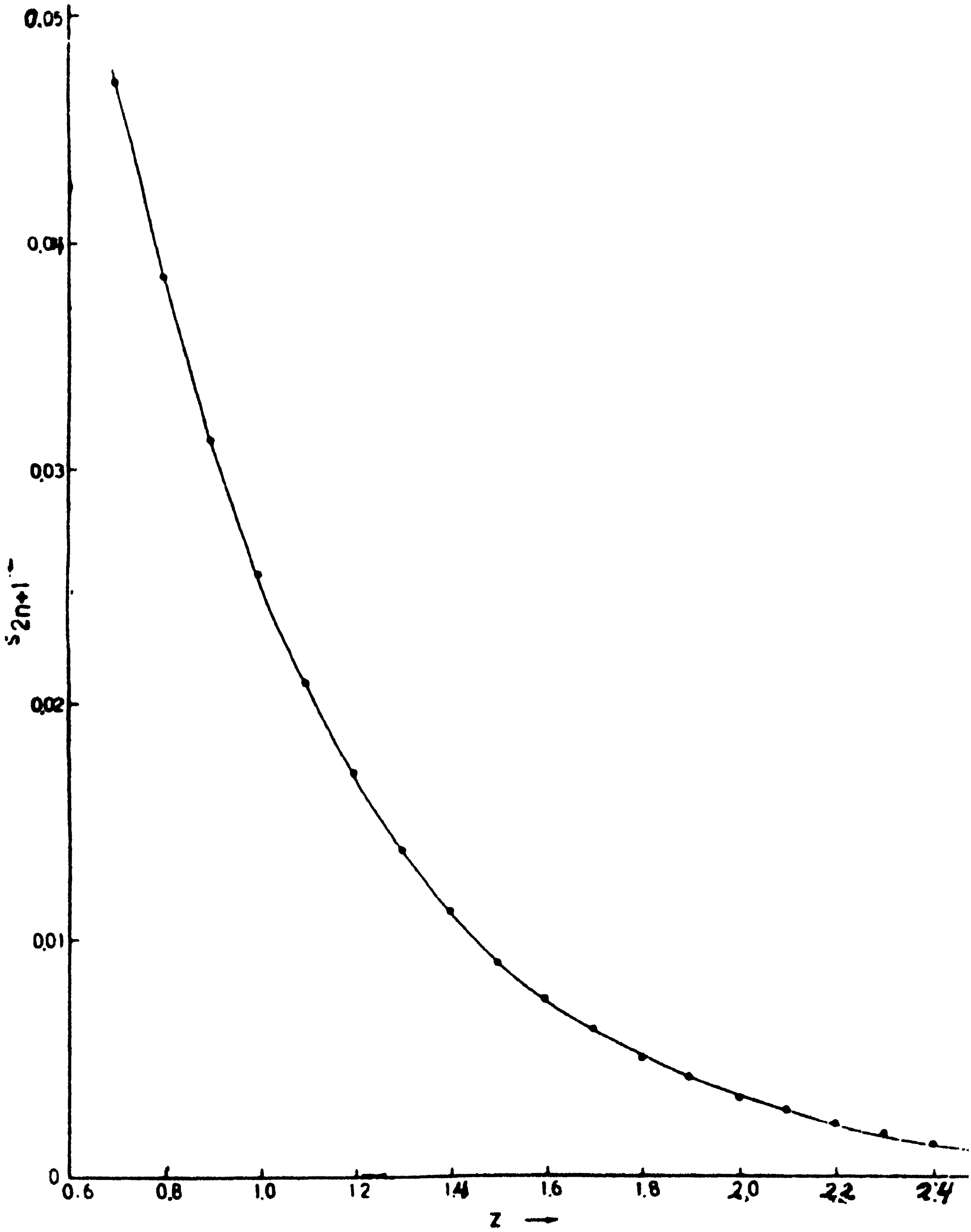

Figure 1. Diatomic Toda chain (oddsite s_{2n+1} *versus* z).

can not give accurate results in zero region. So we only use those expressions to guess the form of diatomic solution. The graphs exhibit identical form as the equal mass case, but the evensite excitations appear to be stronger than odd-site excitations. This analysis unexpectedly, suggests soliton modes in acoustic region only.

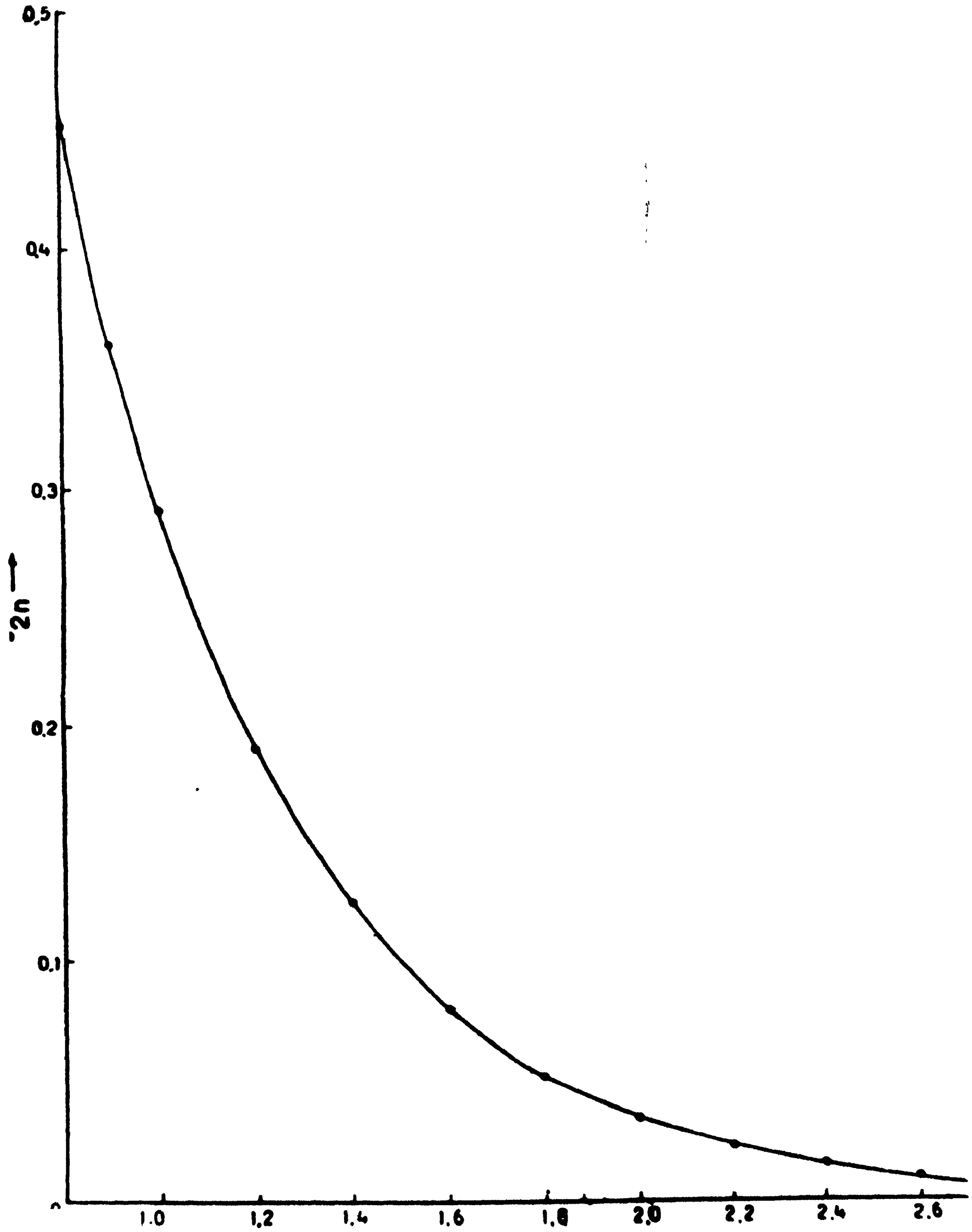

Figure 2. Diatomic Toda chain (evensite s_{2n} *versus* z).

4. Conclusions

We have succeeded in applying a heuristic physical approach, the mixing exponential method to a differential difference equation. Our analysis on Toda lattice equation indicates that for tackling discrete systems it is not convenient to go over to travelling co-ordinate system, instead a series in exponentials is to be directly applied and necessary mathematical identities should be used or established, if necessary. We have established one such identity in the appendix. The general series solution for the single soliton case is obtained in eqs. (7) and (15). The choice of parameters are made in section **2b** to obtain all known single soliton solutions of Toda lattice equation. The present work may be useful for finding nonlinear solitary wave solutions for many interesting discrete models which are now being studied in the continuum limit because of the absence of an easily accessible method to solve the discrete counterpart. Another use of this approach may be its application to the controversial diatomic Toda model. We analysed this case in Section 3.

On the whole, the purpose of this paper is two fold. *First,* the mixing exponential method is expected to bring out a general recipe for obtaining solitary wave solutions in discrete and continuous cases, in complete integrable as well as partially integrable cases in single fields and coupled fields and the like. The very forceful methods like IST, Hirota and Backlund are applicable for complete integrable equations and fail to deliver results for partially integrable cases. Since in the literature all cases discussed using mixing exponential are continuum situations, we analysed a discrete difference equation to illustrate its general nature. However, our analysis shows that discrete cases are extremely difficult because it requires establishment of new mathematical identities which we have succeeded in finding for monatomic case. In this discrete case, even the next complex problem of finding two soliton solutions becomes unmanagably difficult due to lack of necessary mathematical identity and till to-day we fail to establish such identity. Hence our conclusion in this regard is that mixing exponential method as it stands now, can be considered as a general approach for obtaining single soliton solutions. The second purpose of this paper is to shed new light on the diatomic Toda case for which soliton modes may exist as shown in some numerical calculations [7] and so analytical methods should suggest its presence. This was done approximately by mixing exponential method as has been shown in Section 3.

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Appendix

By algebraic simplification we may write :

$$-2^2+[\exp(2Kx)+\exp(-2Kx)-2][\exp(Kx)+\exp(-Kx)-2)]^{-1}$$

$$=\ (2-1)[\exp(Kx)+\exp(-Kx)-2]$$

$$-3^2+[\exp(3Kx)+\exp(-3Kx)-2][\exp(Kx)+\exp(-Kx)-2]^{-1}$$

$$=\ (3-1)[\exp(Kx)+\exp(-Kx)-2]+(3-2)[\exp(2Kx)+\exp(-2Kx)-2]$$

and

$$-4^2+[\exp(4Kx)+\exp(-4Kx)-2][\exp(Kx)+\exp(-Kx)-2]^{-1}$$

$$=\ (4-1)[\exp(Kx)+\exp(-Kx)-2]+(4-2)[\exp(2Kx)+\exp(-2Kx)-2]$$

$$+(4-3)[\exp(3Kx)+\exp(-3Kx)-2].$$

Using the method of induction, the general form becomes transparent :

$$-j^2+[\exp(jKx)+\exp(-jKx)-2][\exp(Kx)+\exp(-Kx)-2]^{-1}$$

$$=\sum_{m=1}^{j-1}(j-m)[\exp(mKx)+\exp(-mKx)-2]. \qquad (A_1)$$

The general identity (A_1) can be verified as follows :

$$\text{R.H.S.} = (j-1)(e^{Kx}+e^{-Kx}-2)+(j-2)(e^{2Kx}+e^{-2Kx}-2)+\ldots$$

$$\ldots+[j-(j-1)][e^{(j-1)kx}+e^{-(j-1)Kx}-2]$$

$$=\ +j(a+a^2+a^3+\ldots\ldots+a^{j-1})-\{a+2a^2+3a^3+\ldots+(j-1)a^{j-1}$$

$$+j(b+b^2+b^3+\ldots\ldots+b^{j-1})-\{b+2b^2+3b^3+\ldots+(j-1)b^{j-1}$$

$$-2[1+2+3+\ldots\ldots(j-1)],\ a\equiv e^{Kx} \text{ and } b=e^{-Kx}$$

$$=j\frac{a(1-a^{j-1})}{1-a}-\frac{a(1-a^{j-1})}{(1-a)^2}+\frac{(j-1)a^j}{1-a}+j\frac{b(1-b^{j-1})}{1-b}$$

$$-\frac{b(1-b^{j-1})}{(1-b)^2}+\frac{(j-1)b^j}{1-b}-2\frac{j(j-1)}{2}$$

$$=\ -j^2+j+\frac{(j-1)a-ja^2+a^{j+1}}{(1-a)^2}+\frac{(j-1)b-jb^2+b^{j+1}}{(1-b)^2}$$

$$= -j^2 + j + \frac{(j-1) - ja + a^j}{(a^{-1/2} - a^{1/2})^2} + \frac{(j-1) - jb + b^j}{(b^{-1/2} - b^{1/2})^2}$$

$$= -j^2 + j + \frac{2(j-1) - ja - jb + a^j + b^j}{e^{kx} + e^{-kx} - 2}$$

$$= -j^2 + j + \frac{-j(a+b-2) + a^j + b^j - 2}{a+b-2}$$

$$= -j^2 + j - j + \frac{a^j + b^j - 2}{a+b-2}$$

$$= -j^2 + \frac{e^{jKx} + e^{-jKx} - 2}{e^{Kx} + e^{-Kx} - 2} \equiv \text{L.H.S.}$$